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KAZHDAN GROUPS WITH INFINITE OUTER AUTOMORPHISM GROUP

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Abstract. For each countable group Q we produce a short exact sequence $1 \to N \to G \to Q \to 1$ where G is f.g. and has a graphical $\frac{1}{6}$ presentation and N is f.g. and satisfies property T.

As a consequence we produce a group N with property T such that $\mathrm{Out}(N)$ is infinite.

Using the tools developed we are also able to produce examples of nonHopfian and non-coHopfian groups with property T.

One of our main tools is the use of random groups to achieve certain properties.

1. Introduction

The main result of this paper is a variant of Rips' construction which allows us to get groups with infinite outer automorphism group, combined with a tool of Gromov to get property T at the same time. Yet other variants provide non-Hopfian and non-coHopfian groups with property T.

In [Rip82], Rips gave an elementary construction which given a countable group Q produces a short exact sequence $1 \to N \to G \to Q \to 1$, where G is a $C'(\frac{1}{6})$ group and N is finitely generated. Rips used his construction to produce $C'(\frac{1}{6})$ presentations with various interesting properties, by lifting pathologies in Q to suitably reinterpreted pathologies in G.

Besides, Gromov [Gro03] was able to produce (random) groups with property T having so-called *graphical* $1/6$ *small cancellation* (or $Gr'(\frac{1}{6})$ for short) presentation, which is a kind of generalized $C'(\frac{1}{6})$ small cancellation property (see section 2 below).

The mixture of these two tools yields the following (section 3):

Theorem 1.1. *For each countable group Q, there is a short exact sequence $1 \to N \to G \to Q \to 1$ such that*

(1) *G is torsion-free,*
(2) *G has a graphical $\frac{1}{6}$ presentation, and*
(3) *N has property T.*

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(4) *Moreover, G is finitely generated if Q is, and finitely presented if Q is.*

The graphical 1/6 presentation keeps enough properties of ordinary small cancellation as to mix nicely with Rips' construction. However, we note that Theorem 1.1 cannot be obtained with G an ordinary $C'(\frac{1}{6})$ group, since finitely presented $C'(\frac{1}{6})$ groups act properly on a $CAT(0)$ cube complex by [Wis04], and hence their infinite subgroups cannot have Property T [NR97, NR98].

We apply Theorem 1.1 to obtain the following (section 4):

Theorem 1.2. *There exists a group N with property T such that* $\mathrm{Out}(N)$ *is infinite.*

In fact we even prove that any countable group embeds in $\mathrm{Out}(N)$ for some Kazhdan group N.

The motivation is that, as proven by Paulin [Pau91], if H is word-hyperbolic and $|\operatorname{Out}(H)| = \infty$ then H splits over an infinite cyclic group, and hence H cannot have property T. The question of whether every group with property T has a finite outer automorphism group belongs to the list of open problems mentioned in de la Harpe and Valette's classical book on Property T ([dlHV89], p. 134), was raised again by Alain Valette in his mathscinet review of [Pau91], and later appeared in a problem list from the 2002 meeting on property T at Oberwolfach.

Finally, we use the tools we developed to obtain the following two examples (sections 6 and 5):

Theorem 1.3. *There exists a Kazhdan group G that is not Hopfian.*

Theorem 1.4. *There exists a Kazhdan group G that is not coHopfian.*

Various other attempts to augment Rips's construction have focused on strengthening the properties of G when Q is f.p. (e.g.: G is π_1 of a negatively curved complex [Wis98]; G is a residually finite $C'(\frac{1}{6})$ group [Wis]; G is a subgroup of a right-angled Artin group, so $G \subset SL_n(\mathbb{Z})$ [HW04]).

One key ingredient of our constructions is the use of random methods, introduced by Gromov [Gro93] (see also [Ghy03] and [Oll04] for a discussion of random groups), to provide examples of groups with particular properties. Namely, we use a result of [Gro03] providing a presentation of a group with property T satisfying the graphical small cancellation property. We include in section 7 a standalone proof of the results we need from [Gro03].

2. $Gr'(\frac{1}{6})$ GRAPHS

2.1. **Review of graphical α-condition $Gr'(\alpha)$.** Throughout all this article, B is a bouquet of $m \geqslant 2$ circles whose edges are directed and labelled, so that m will be the number of generators of the group presentatations we consider.

Let $\Gamma \looparrowright B$ be an immersed graph, and note that Γ has an induced labelling. That Γ immerses in B simply denotes the fact that the words carried by paths immersed in Γ are reduced.

By definition, the group G presented by $\langle B|\Gamma\rangle$, has generators the letters appearing on B, and relations consisting of all cycles appearing in F.

A *piece* P in Γ is an immersed path $P \looparrowright B$ which lifts to Γ in more than one way.

Definition 2.1. We say $\Gamma \looparrowright B$ satisfies the *graphical α condition $Gr'(\alpha)$* if for each piece P, and each cycle $C \to \Gamma$ such that $P \to \Gamma$ factors through $P \to C \to \Gamma$, we have $|P| < \alpha|C|$.

The graphical α condition generalizes the usual $C'(\alpha)$: let F consist of the disjoint union of a set of cycles corresponding to the relators in a presentation. The graphical α condition is a case of a complicated but more general condition given by Gromov [Gro03].

The condition $Gr'(\frac{1}{6})$ implies that the group G is torsion-free, word-hyperbolic, of dimension 2, just as the $C'(1/6)$ condition [Oll03b]. The group is non-elementary except in some explicit degenerate cases (a hyperbolic group is called elementary if it is finite or virtually $\mathbb{Z}$).

There is also a slightly stronger version of this condition, in which we demand that the size of the pieces be bounded not by α times the size of any cycle containing the piece, but by α times the girth of Γ (recall the *girth* of a graph is the smallest length of a non-trivial closed path in it). We will sometimes directly prove this stronger version below, since it allows lighter notations.

A *disc van Kampen diagram* w.r.t. a graphical presentation is a van Kampen diagram every 2-cell of which is labelled by a closed path immersed in Γ. It is *reduced* if, first, it is reduced in the ordinary sense and if moreover, for any two adjacent 2-cells, the boundary word of their union does not embed as a closed path in Γ (otherwise, these two 2-cells can be replaced by a single one). It is proven in [Oll03b] that if $Gr'(\alpha)$ holds, such a reduced van Kampen diagram satisfies the ordinary $C'(\alpha)$ condition.

2.2. **Producing more $Gr'(\frac{1}{6})$ graphs.**

One useful feature of a presentation satisfying the ordinary $C'(\frac{1}{6})$ theory is that, provided that the relations are not "too dense" in a certain sense, more relations can be added to the presentation without violating the $C'(\frac{1}{6})$ condition.

In this subsection, we describe conditions on a $Gr'(\alpha)$ presentation such that additional relations can be added.

Proposition 2.2. *Let $\Gamma \looparrowright B$ satisfy the $Gr'(\alpha)$ condition and suppose there is an immersed path $W \to B$ such that $1 \leqslant |W| < \frac{\alpha}{2}\,\mathrm{girth}(\Gamma) - 1$, and W does not lift to Γ.*

Then there is a set of closed immersed paths $C_i \looparrowright B : i \in \mathbb{N}$ such that the disjoint union $\Gamma' = \Gamma \sqcup_{i\in\mathbb{N}} C_i \looparrowright B$ satisfies the $Gr'(\alpha)$ condition.

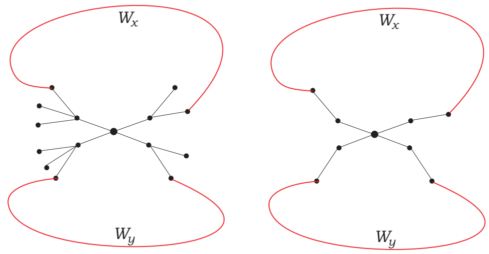

FIGURE 1.

Proof. We first form an immersed labelled graph $A \looparrowright B$ as follows: Let D be the radius 2 ball at the basepoint of the universal cover $\tilde{B}$, and attach two copies W_x and W_y of the arc W along four distinct leaves of D as in Figure 1. (This can always be done avoiding the inverses of the initial and final letter of W, so that D immerses in B). Finally, we remove the finite trees that remain.

Observe that any path $P \looparrowright B$ that lifts to both A and Γ satisfies $|P| < \alpha\,\mathrm{girth}(\Gamma)$. Indeed, if P lifts to Γ, then P cannot contain W_x or W_y as a subpath, and hence $P = U_1U_2U_3$ where U_1 and U_3 are proper initial or terminal subpaths of a W-arc, and U_2 is a path in D, so $|P| \leqslant |U_1| + |U_2| + |U_3| \leqslant (|W| - 1) + 4 + (|W| - 1) = 2|W| + 2 = 2(|W| + 1) < 2\frac{\alpha}{2}\,\mathrm{girth}(\Gamma) = \alpha\,\mathrm{girth}(\Gamma)$.

Now let x and y be arbitrary labels. To any reduced word w in the letters $x^{\pm1}$ and $y^{\pm1}$ we can associate an immersed closed path $\varphi(w)$ in A by sending x to the based path in A containing W_x, and similarly for y.

Now for each $i \in \mathbb{N}$, let c_i denote the word $xy^{1000i+1}xy^{1000i+2}\cdots xy^{1000i+999}$. It is easily verified that for large enough values of 1000, the set of words $\langle x, y \mid c_i : i \in \mathbb{N}\rangle$ satisfies the $C'(\frac{\alpha}{2})$ condition.

Let $C_i \looparrowright B$ denote the corresponding closed immersed cycle $\varphi(c_i)$. Pieces in $\bigsqcup C_i$ are easily bounded in terms of pieces in $\langle x, y \mid c_i\ (i \in \mathbb{N})\rangle$, so that $\bigsqcup C_i$ satisfies the $Gr'(\alpha)$ (actually $C'(\alpha)$) condition.

Finally $\Gamma' = \Gamma \sqcup_{i\in\mathbb{N}} C_i$ satisfies the $Gr'(\alpha)$ condition since pieces that lift twice to Γ are bounded by assumption, and we have just bounded pieces that lift to Γ and to some C_i, and pieces that lift to some C_i and some C_j. $\square$

Remark 2.3. The missing word condition in Γ ensures that the group presented by Γ is non-elementary. Indeed, the group presented by $\Gamma \sqcup_i C_i$ has infinite Euler characteristic (it is of dimension 2) and is thus non-elementary, so a fortiori the group presented by Γ is.

3. The T Rips construction

Let us now turn to the proof of the main theorem of this article. We use an intermediate construction due to Gromov.

Proposition 3.1. *There exists a finite graph Γ that immerses in a bouquet B of two circles such that:*

1. *The group presented by $\langle B \mid \Gamma\rangle$ has property T.*
2. *$\Gamma \looparrowright B$ satisfies the $Gr'(\frac{1}{12})$ condition.*
3. *There is a path $W \looparrowright B$ with $1 \leqslant |W| < \frac{1}{24}\operatorname{girth}(\Gamma) - 1$ and W does not lift to Γ.*
4. *Γ has arbitrarily large girth.*

A proof of this is included below (section 7).

Theorem 1.1. *For each countable group Q, there is a short exact sequence $1 \to N \to G \to Q \to 1$ such that*

1. *G is torsion-free,*
2. *G has a graphical $\frac{1}{6}$ presentation, and*
3. *N has property T and is non-trivial.*
4. *Moreover, G is finitely generated if Q is, and finitely presented if Q is.*

Proof. Let Q be given by the following presentation:

$$\langle q_i : i \in I \mid R_j : j \in J\rangle$$

Let $\Gamma \looparrowright B$ be a graph provided by Proposition 3.1, where the edges of B are labelled by x and y. Let $\Gamma' = \Gamma \sqcup_n C_n$ be as in Proposition 2.2 with $\alpha = 1/12$.

The presentation for G will be the following:

$$(1) \quad \langle\ x, y,\ q_i\ (i \in I)\ \mid\ \Gamma,$$
$$x^{q_i} = X_{i+},\ x^{q_i^{-1}} = X_{i-},\ y^{q_i} = Y_{i+},\ y^{q_i^{-1}} = Y_{i-}\ (i \in I),$$
$$R_j = W_j\ (j \in J)\ \rangle$$

where superscripts denote conjugation, and where the X_{i+}, X_{i-}, Y_{i+}, Y_{i-}, and W_j are equal to paths corresponding to distinct C_n cycles of Γ', $|W_j| > 12|R_j|$ for each $j \in J$, and $|X_{i\pm}| > 36, |Y_{i\pm}| > 36$ for each $i \in I$.

The $\frac{1}{6}$ condition follows easily. Let us check, for example, that there is no $\frac{1}{6}$-piece between Γ and the relation $x^{q_i} = X_{i+}$. Since the q_i's do not appear as labels on Γ, any such $\frac{1}{6}$-piece would be either x or a subword of X_{i+}. The former is ruled out since $\operatorname{girth}(\Gamma) > 6$. The latter would provide a piece between Γ and X_{i+} (which is one of the C_n's); such a piece is by assumption of length at most $\frac{1}{12}|X_{i+}|$ which in turn is less than $\frac{1}{6}|x^{q_i} = X_{i+}|$ as needed. The other cases are treated similarly.

Now N is the subgroup of G generated by x and y. It is normal by construction of the presentation of G. Note that N has property T since it is a quotient of $\langle x, y \mid \Gamma\rangle$ which has property T by choice of Γ.

Finally, N is non-trivial: indeed, we can pick some cycle C_n which is a word in x, y and which will be in small cancellation with the rest of the

presentation. This provides a word in x and y whichh is not trivial in the group. □

4. Kazhdan groups with infinite outer automorphism group

Theorem 1.2. *Any countable infinite group Q embeds in $\mathrm{Out}(N)$ for some group N with property T.*

In particular, there exists a group N with property T such that $\mathrm{Out}(N)$ is infinite.

Proof. For $1 \to N \to G \to Q \to 1$, the group G acts by inner automorphisms on itself, so we have a homomorphism $G \to \mathrm{Aut}(N)$, and N obviously maps to $\mathrm{Inn}(N)$ so there is an induced homomorphism $Q = G/N \to \mathrm{Out}(N)$. Elements in the kernel of $Q \to \mathrm{Out}(N)$ are represented by elements $g \in G$ such that $m^g = m^n$ for some $n \in N$ and all $m \in N$. Thus gn^{-1} centralizes N.

First suppose that Q is finitely presented, so that G is as well.

In this case N is a non-elementary subgroup of the torsion-free word-hyperbolic group G, and hence N has a trivial centralizer. Indeed, N must contain a rank 2 free subgroup $\langle n_1, n_2\rangle$ (see [GdlH90], p. 157). If a nontrivial element c centralizes N then $\langle c, n_1\rangle$ and $\langle c, n_2\rangle$ are both abelian, and hence infinite cyclic since G cannot contain a copy of $\mathbb{Z}^2$. Thus $n_1^{m_1} = c^{p_1}$ and $n_2^{m_2} = c^{p_2}$ for some $p_i, m_i \neq 0$. But then $n_1^{m_1}$ commutes with $n_1^{m_2}$ which is impossible.

Since the centralizer of N is trivial, we have $gn^{-1} = 1$, so $g \in N$, and hence $Q \to \mathrm{Out}(N)$ is injective.

The case when Q is not finitely presented reduces back to the previous one: Indeed, suppose that some element g of G lies in the centralizer of N. This is equivalent to stating that g commutes with x and y. But g can be written as a product of finitely many generators, and similarly the relations $[g, x] = 1$ and $[g, y] = 1$ are consequences of only finitely many relators, so that g still lies in the centralizer of N in a finite subpresentation of the presentation of G. □

Remark 4.1. By adding some additional relations to N, the above argument was used in [BW02] to show that every countable group Q appears as $\mathrm{Out}(N)$ for some f.g. N, and that every f.p. Q appears as $\mathrm{Out}(N)$ where N is f.g. and residually finite (but property T did not appear there).

It appears likely that a more careful analysis along those lines, would show that every countable group arises as $\mathrm{Out}(N)$ where N has property T.

5. A Kazhdan group that is not coHopfian

Theorem 1.4. *There exists a Kazhdan group that is not coHopfian.*

Proof. Consider the group

$$G = \langle\, a, b, t \mid \Gamma,\ a^t = \varphi(a),\ b^t = \varphi(b)\,\rangle$$

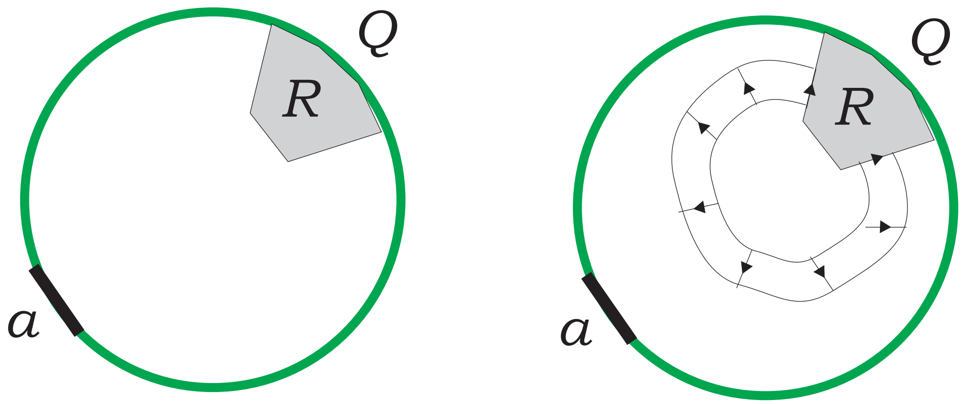

FIGURE 2.

where $\varphi(a)$ and $\varphi(b)$ are chosen so that $\Gamma \sqcup \varphi(a) \sqcup \varphi(b)$ satisfies $Gr'(\frac{1}{6})$ and $|\varphi(a)| > 3$, $|\varphi(b)| > 3$. (This is in fact a subpresentation of the presentation (1) used in the proof of Theorem 1.1.)

Clearly, the subgroup $K = \langle a, b\rangle$ is a Kazhdan group since it is a quotient of $\langle a, b \mid \Gamma\rangle$.

The map $K \to K$ induced by φ is clearly well-defined and injective since it arises from conjugation in the larger group G.

We will now show that φ is not surjective by verifying that $a \notin \langle\varphi(a), \varphi(b)\rangle$.

We argue by contradiction: Suppose that a is equal in G to a word $W(\varphi(a), \varphi(b))$ in $\varphi(a)$ and $\varphi(b)$; we can choose W such that the disc diagram expressing this equality in the presentation for G has minimal area among all such choices. Note that since D is reduced and G is $Gr'(\frac{1}{6})$, D is a diagram satisfying the ordinary $C'(\frac{1}{6})$ condition.

By Greendlinger's Lemma, (after ignoring trees possibly attached to ∂D) either D is a single 2-cell, or D has at least two 2-cells whose outer paths are the majority of their boundaries.

The first possibility is excluded by consideration of the presentation for G. In the second case, one such 2-cell R has outerpath Q not containing the special a-edge in ∂D, as illustrated on the left in Figure 2.

The boundary word of 2-cell R cannot be a words immersing in Γ. Indeed, since it has more than half its length on the boundary of D and this boundary bears a word in $\varphi(a)$ and $\varphi(b)$, this would contradict the small cancellation property of $\Gamma \sqcup \varphi(a) \sqcup \varphi(b)$. So R is a 2-cell expressing the equality $a^t = \varphi(a)$ or $b^t = \varphi(b)$. Moreover, since t does not appear on the boundary of D, the side of R on the boundary is the φ-side.

Since $t \notin \partial D$, we can find a t-annulus containing R as illustrated in the center of in Figure 2.

We now produce a new diagram D' with $\mathsf{Area}(D') < \mathsf{Area}(D)$. We do this by travelling around the t-annulus as on the right in Figure 2.

Observe that the small cancellation property implies that an edge in the $\varphi(a) \subset \partial R$ or $\varphi(b) \subset \partial R$ lines up with an edge in some $\varphi(a)$ or $\varphi(b)$ in ∂D, and at exactly the same position. So if the $\varphi(a)$ or $\varphi(b)$ of ∂R is not wholly contained in ∂D, after removing R the words on the paths from ∂D to the t-edges of R will cancel with corresponding subwords of $\varphi(a)$ and $\varphi(b)$ lying in the remaining part of ∂D.

This implies that, after removing the annulus, the boundary of D' is labelled (maybe after folding) by a word of the form $a = W'(\varphi(a), \varphi(b))$. But this is a contradiction since D was assumed to be minimal.

(Note that D' might touch the special a-edge, and D' might have some extra singular edges.) □

6. A Kazhdan group that is not Hopfian

Definition 6.1. Let B be a bouquet of circles, and let $\varphi : B \to B$. Let $A \to B$ be a map of graphs, then we let $\varphi(A) \to B$ be the new map of graphs where $\varphi(A)$ is obtained from A by substituting an arc $\varphi(e)$ for each edge e of A. That is, we replace the label on each edge of A by its image under φ.

Lemma 6.2. *Let Δ be a labelled graph satisfying $Gr'(\alpha)$ and $\alpha\,\mathrm{girth}(\Delta) \geqslant 1$. Suppose there is a path $P \looparrowright \Delta$ such that the edges in P all bear the same label a and such that P factors through a closed path $P \looparrowright C \looparrowright \Delta$. Then $|P| < 2\alpha\,|C|$.*

Note that the assumption $\mathrm{girth}(\Delta) \geqslant 1/\alpha$ is not very strong: if $\alpha\,\mathrm{girth}(\Delta) \leqslant 1$ then a single letter can constitute a piece, which can result in various oddities. This lemma is false for trivial reasons if we remove this girth assumption: when $\mathrm{girth}(\Delta) = 1$ there are arbitrarily long homogeneous paths though $Gr'(0)$ may be satisfied.

Proof. First, let us treat the trivial case when there is a length-1 loop bearing label a: this implies $\mathrm{girth}(\Delta) = 1$ so $\alpha = 1$ and the equality to show is trivial. The case $|P| = 1$ is trivial as well.

Second, suppose that there is no length-1 loop. Let P be a path labelled by a^s with $s \geqslant 2$. Then the two paths labelled by a^{s-1} obtained by removing the first and last edge of P respectively constitute a piece, and so we have $s - 1 < \alpha\,|C|$ so that $|P| = s < \alpha\,|C| + 1 \leqslant \alpha(|C| + \mathrm{girth}(\Delta)) \leqslant 2\alpha\,|C|$. □

Lemma 6.3. *Let Δ be a labelled graph satisfying $Gr'(\alpha)$ with $\alpha\,\mathrm{girth}(\Delta) \geqslant 1$, and let $\varphi : B \to B$ be induced by $a \mapsto a^n$ and $b \mapsto b^n$ for some $n \geqslant 1$. Then, for any $k \in \mathbb{N}$, $\varphi^k(\Delta)$ satisfies $Gr'(2\alpha)$.*

(Once more the girth assumption discards some degenerate cases when a single edge can make a piece.)

Proof. The reader should think of $\varphi^k(\Delta)$ as the n^k-subdivision of Δ where each a-edge is replaced by an arc of n^k a-edges and likewise for b-edges.

We begin by considering a homogeneous piece $P = a^r$ (or $P = b^r$ which is similar) occurring in some cycle $C \looparrowright \varphi^k(\Delta)$. Then P is a subpath of a path $\varphi^k(P')$ where $P' = a^{r'}$ is a path in Δ and P' occurs in a cycle C' corresponding to C.

By the previous lemma, $|P'| = r' < 2\alpha|C'|$ and so $|P| = r < 2\alpha n^k |C'| = 2\alpha|C|$.

We now consider the general case where P contains both a and b letters. We may assume that P is a maximal piece, in which case $P = W(a^{n^k}, b^{n^k})$ where $P' = W(a, b)$ is itself a corresponding piece in Δ. Everything scales by n^k i.e. $|P| = n^k|P'| < n^k\alpha|C'| = \alpha|C|$. □

Lemma 6.4. *Let Δ satisfy $Gr'(\alpha)$ and suppose that* $\operatorname{girth}(\Delta) > 1/\alpha$. *Let n satisfy $n > s$ where s is the maximal length of a path a^s or b^s lifting to Δ. Let $\varphi : B \to B$ be induced by $a \mapsto a^n$ and $b \mapsto b^n$.*

Then $\bigsqcup_{k\geqslant 0} \varphi^k(\Delta)$ satisfies $Gr'(8\alpha)$.

Note that $s = \infty$ implies either $\operatorname{girth}(\Delta) = 1$ (which is excluded by assumption) or $\alpha = 1$ (by removing the first and last letter of an arbitrarily long a^s-path) in which case the affirmation is void. So we can suppose $s < \infty$.

Proof. First, by the previous lemma, each $\varphi^k(\Delta)$ itself satisfies $Gr'(2\alpha)$.

We now consider a piece P between Δ and $\varphi^k(\Delta)$. Either $P \looparrowright \varphi^k(\Delta)$ is contained in two subdivided edges of $\varphi^k(\Delta)$ so $|P| < 2n^k$; or P contains an entire subdivided edge and hence an a^{n^k} (or b^{n^k}) subpath.

In the latter case when P contains an a^{n^k} or b^{n^k} subpath, since $P \looparrowright \Delta$ is a path in Δ then n^k is at most the maximal length of an a-path or b-path in Δ. But by hypothesis on n, this maximal length is bounded by n, and so $n^k < n$ which is impossible for $k \geqslant 1$.

In the former case, P is the product of at most two homogeneous paths (i.e. a-paths or b-paths) one of which has length $\geqslant \frac{1}{2}|P|$. Thus by Lemma 6.2, $\frac{1}{2}|P| < 2\alpha|C|$ for any cycle C in Δ containing P. So $|P| < 4\alpha|C|$ and so P cannot be a 4α-piece in Δ. Besides, suppose that P is included in a cycle C immersed in $\varphi^k(\Delta)$. Since $|P| < 2n^k$ and $|C| \geqslant \operatorname{girth} \varphi^k(\Delta) = n^k \operatorname{girth}(\Delta) \geqslant n^k/\alpha$ by assumption, P cannot consitute a 2α-piece in $\varphi^k(\Delta)$ either. (Note that the constant 4 is this reasoning is optimal: consider for Δ a circle of length 100 containing $aabb$ at one place, and some garbage for the rest; take $\alpha = (1+\varepsilon)/100$ so that the two a's do not form a piece. Then $\varphi(\Delta)$ contains some $aabb$ as well, so that this word constitutes a $4/100$-piece in $\Delta \sqcup \varphi(\Delta)$.)

Finally, we consider pieces between $\varphi^k(\Delta)$ and $\varphi^{k'}(\Delta)$ where we can suppose $k' > k$. We have just proved that $\Delta \sqcup \varphi^{k'-k}(\Delta)$ satisfies $Gr'(4\alpha)$. We now apply Lemma 6.3 to see that $\varphi^k(\Delta) \sqcup \varphi^{k'}(\Delta) = \varphi^k\left(\Delta \sqcup \varphi^{k'-k}(\Delta)\right)$ satisfies $Gr'(8\alpha)$. □

Remark 6.5. A generalization of Lemma 6.4 should hold with $\varphi(a)$ and $\varphi(b)$ appropriate small cancellation words instead of a^n and b^n.

Theorem 1.3. *There exists a Kazhdan group that is not Hopfian.*

Proof. Let G have the following presentation:

$$\langle\, a, b \mid \varphi^i(\Gamma),\ \varphi^i(a\varphi(C_1)),\ \varphi^i(b\varphi(C_2)),\ \varphi^i(\varphi(C_3))\ (i \geqslant 0)\,\rangle$$

where

1. $\Gamma \sqcup C_1 \sqcup C_2 \sqcup C_3$ satisfies the $Gr'(\alpha)$ condition with $\alpha = 1/2000$ (C_1, C_2 and C_3 arise from Proposition 2.2);
2. φ is defined by $\varphi(a) = a^n$ and $\varphi(b) = b^n$, for some n greater than the maximal length of an a-word or b-word in $\Gamma \sqcup C_1 \sqcup C_2 \sqcup C_3$;
3. $\operatorname{girth}(\Gamma \sqcup C_1 \sqcup C_2 \sqcup C_3) \geqslant 2000$.

Let $\Delta_0 = \bigsqcup_{k\geqslant 0} \varphi^k\,(\Gamma \sqcup C_1 \sqcup C_2 \sqcup C_3)$. By Lemma 6.4, this labelled graph satisfies $Gr'(8\alpha)$. As a subgraph of Δ_0, the graph $\Delta = \Gamma \sqcup \varphi(C_1) \sqcup \varphi(C_2) \sqcup C_3$ satisfies $Gr'(8\alpha)$ as well.

We now prove that $\Delta' = \Gamma \sqcup a\varphi(C_1) \sqcup b\varphi(C_2) \sqcup C_3$ is $Gr'(26\alpha)$. Let P be a piece involving the new a-edge or the new b-edge. Observe that $P = P_1 a P_2$ (or $P_1 b P_2$). Note that a new b (or new a) may lie in at most one of of P_1 or P_2. Thus P is the concatenation of at most 3 pieces in Δ together with the new a and possibly the new b. Consequently for any cycle C containing P we have $|P| < 24\alpha|C| + 2 \leqslant 26\alpha|C'|$ where we have used the hypothesis that $\alpha \operatorname{girth} \geqslant 1$.

We now apply Lemma 6.4 to see that $\Omega = \bigsqcup_{k\geqslant 0} \varphi^k(\Delta')$ satisfies $Gr'(208\alpha)$, and so does the presentation for G which is a subset of Ω.

Now φ obviously sends relations to relations and thus induces a well-defined map in G. This map is surjective since $a\varphi(C_1) =_G 1$ and $b\varphi(C_2) =_G 1$.

Finally φ is not injective since $\varphi(C_3) =_G 1$ but $C_3 \neq_G 1$. Indeed, C_3 is in small cancellation relative to the relators of G since both are included in Ω. □

7. A T $Gr'(\frac{1}{6})$ graph with a missing word

A main point in this paper is the following, introduced by Gromov in [Gro03]:

Proposition 7.1. *For each $\alpha > 0$ and $\alpha' > 0$ there exists a finite graph Γ that immerses in a bouquet B of two circles such that:*

1. *The group presented by $\langle\, B \mid \Gamma\,\rangle$ has property T.*
2. *$\Gamma \looparrowright B$ satisfies the $Gr'(\alpha)$ condition.*
3. *There is a path $W \looparrowright B$ with $1 \leqslant |W| \leqslant \alpha' \operatorname{girth}(\Gamma)$ and W does not lift to Γ.*

Moreover, the girth of Γ can be taken arbitrarily large.

This trivially implies Proposition 3.1. It also results from Remark 2.3 that the obtained group is non-trivial.

The goal of the introduction of such graphs in [Gro03] was to construct a group whose Cayley graph contains a family of expanders, in relation with the Baum-Connes conjecture (see also [Ghy03] and [Oll03a]). There,

the construction is done starting not only with a free group but with an arbitrary hyperbolic group (compare [Oll04]), so that it can be iterated in order to embed a whole family of graphs.

Here we use this construction for purposes closer to combinatorial group theory. We do not need the full strength of the iterated construction; this section is devoted to the proof of the statements we need.

We will use the following fact, the credit of which can be shared between Lubotzky, Margulis, Phillips, Sarnack, Selberg. We refer to [Lub94] (Theorem 7.4.4 referring to Theorem 7.3.12), or to to [DSV03].

Proposition 7.2. *For lost of $v \in \mathbb{N}$, there is a family of graphs $\Gamma_i : i \in \mathbb{N}$ such that the following hold:*

(1) *Each Γ_i is regular of valence v.*
(2) $\inf_i \lambda_1(\Gamma_i) > 0$ *where λ_1 denotes the smallest non-zero eigenvalue of the discrete Laplacian Δ.*
(3) $\operatorname{girth}(\Gamma_i) \longrightarrow \infty$.
(4) $\exists C$ *such that* $\operatorname{Diameter}(\Gamma_i) \leqslant C \operatorname{girth}(\Gamma_i)$ *for all i.*

"Lots of v" means e.g. that this works at least for $v = p + 1$ with $p \geqslant 3$ prime ([Lub94], paragraph 1.2 refers to other constructions). This is irrelevant for our purpose.

We are going to use random labellings of subdivisions of the graphs Γ_i. Subdividing amounts to labelling each edge with a long word rather than just one letter, so that the small cancellation condition is more easily satisfied.

That the diameter of the graph is bounded by a constant times the girth reflects the fact that there are "not too many" relations added (compare the density model of random groups in [Gro93] or [Oll04]): this amounts to taking an arbitrarily small density.

To prove Proposition 7.1 we need two more propositions.

Proposition 7.3. *Given $v \in \mathbb{N}$, $\lambda_0 > 0$ and an integer $j \geqslant 1$ there exists an explicit g_0 such that if Γ is a v-regular graph with* $\operatorname{girth}(\Gamma) \geqslant g$, $\lambda_1(\Gamma) \geqslant \lambda_0$ *and Γ is trivalent, then the random group defined through a random labelling of the j-subdivision Γ^j of Γ will have property T, with probability tending to* 1 *as the size of Γ tends to infinity.*

This is proven in [Sil03] (Corollary 3.19 where d is our v, k is our number of generators m, and $|V|$ the size of the graph; in this reference, $\lambda(\Gamma)$ denotes the largest eigenvalue not equal to 1 of the averaging operator $1-\Delta$, so that the inequalities between this λ and the first non-zero eigenvalue of Δ are reversed.)

In the next proposition and for the rest of this section, Γ^j denotes the j-subdivision of (the edges of) the graph Γ.

Proposition 7.4. *For any $v \in \mathbb{N}$, any $\alpha > 0$ and $\alpha' > 0$, for any $C \geqslant 1$, there exists an integer j_0 such that for any $j \geqslant j_0$, for any graph Γ satisfying the conditions:*

(1) *Each vertex of* Γ *is of valence at most* v*;*
(2) *The girth of* Γ *is* g*;*
(3) $\mathrm{Diameter}(\Gamma) \leqslant Cg$ *for all* i*;*

then the following properties hold with probability tending to 1 *as* $g \to \infty$*:*

(1) *The folded graph* $\overline{\Gamma^j}$ *obtained by a random labelling of* Γ^j *satisfies the* $Gr'(\alpha)$ *condition.*
(2) *There is a reduced word of length between* 1 *and* $\alpha' \operatorname{girth} \overline{\Gamma^j}$ *not appearing on any path in* $\overline{\Gamma^j}$*.*

This will be proven in the next sections (a sketch of proof can also be found in [Gro03]).

Let us now just gather propositions 7.2, 7.3 and 7.4.

Proof of Proposition 7.1. Let α be the small cancellation constant to be achieved.

Apply Proposition 7.2 with some $v \in \mathbb{N}$ to get an infinite family of graphs Γ_i; let λ_0 be the lower bound on the spectral gap so obtained, and let C be as in this proposition. Let us denote by $\Gamma_{i(g)}$ the first graph in this family having girth at least g.

For the chosen $\alpha > 0$, let j and g be large enough for the conclusions of Proposition 7.4 to hold when applied to $\Gamma_{i(g)}$. Let g be still large enough (depending on j) so that the conclusions of Proposition 7.3 applied to this j hold. This provides a graph satisfying the three requirements of Proposition 7.1. □

7.1. **Some simple properties of random words.** Recall $m \geqslant 2$ is the number of generators we use. We denote by $\|w\|$ the norm in the free group of the word w, that is, the length of the associated reduced word.

Hereafter θ is the gross cogrowth of the free group (we refer to the paragraph "Growth, cogrowth, and gross cogrowth" in [Oll04] for basic properties). Basically, θ is the infimum of the real numbers so that the number of words of length ℓ which freely reduce to the trivial word is at most $(2m)^{\theta\ell}$ for all $\ell \in \mathbb{N}$. In particular, the probability that a random walk in the free group comes back at its origin at time ℓ is at most $(2m)^{-(1-\theta)\ell}$. Explicitly we have $(2m)^\theta = 2\sqrt{2m-1}$ [Kes59].

We state here some elementary properties having to deal with the behavior of reducing a random word. The first one is pretty intuitive.

Lemma 7.5. *Let* W_ℓ *be a random word of length* ℓ *and let* $\overline{W}_\ell$ *be the associated reduced word. Then the law of* $\overline{W}_\ell$ *knowing its length* $\left|\overline{W}_\ell\right| = \|W_\ell\|$ *is the uniform law on all reduced words of this length.*

Proof of the lemma. The group of automorphisms of the $2m$-regular tree preserving some basepoint acts transitively on the points at a given distance from the basepoint and preserves the law of the random walk beginning at this basepoint. □

The following is proven in [Oll04], Proposition 17.

Lemma 7.6. *Let W_ℓ be a random word of length ℓ. Then, for any $0 \leqslant L \leqslant \ell$ we have*

$$\Pr(\|W_\ell\| \leqslant L) \leqslant (2m)^{-\ell(1-\theta)+\theta L}$$

Note that exponent vanishes for $L = \frac{1-\theta}{\theta}\ell < \ell$ (since $\theta > 1/2$). A slightly different, asymptotically stronger version of this lemma is the following.

Lemma 7.7. *Let W_ℓ be a random word of length ℓ. Then, for any L we have*

$$\Pr(\|W_\ell\| \leqslant L) \leqslant \sqrt{\ell \, \frac{2m}{2m-1}}\,(2m)^{-(1-\theta)\ell}(2m-1)^{L/2}$$

Proof. Let B_ℓ be the ball of radius ℓ centered at e in the free group. Let p_x^ℓ be the probability that $W_\ell = x$. We have

$$\begin{aligned}\mathbb{E}(2m-1)^{-\frac{1}{2}\|W_\ell\|} &= \sum_{x\in B_\ell} p_x^\ell\,(2m-1)^{-\frac{1}{2}\|x\|} \\ &\leqslant \sqrt{\sum_{x\in B_\ell}(p_x^\ell)^2}\,\sqrt{\sum_{x\in B_\ell}(2m-1)^{-\|x\|}}\end{aligned}$$

by the Cauchy-Schwarz inequality. But $\sum_{x\in B_\ell}(p_x^\ell)^2$ is exactly the probability of return to e at time 2ℓ of the random walk (condition by where it is at time ℓ) which is at most $(2m)^{-2(1-\theta)\ell}$. Besides, there are $(2m)(2m-1)^{k-1}$ elements of norm k in B_ℓ, so that $\sum_{x\in B_\ell}(2m-1)^{-\|x\|} = \sum_{0\leqslant k\leqslant \ell}(2m)(2m-1)^{k-1}(2m-1)^{-k} = \ell\frac{2m}{2m-1}$. So we get

$$\mathbb{E}(2m-1)^{-\frac{1}{2}\|W_\ell\|} \leqslant \sqrt{\ell\,\frac{2m}{2m-1}}\,(2m)^{-(1-\theta)\ell}$$

Now we simply apply the Markov inequality

$$\begin{aligned}\Pr(\|W_\ell\| \leqslant L) &= \Pr\left((2m-1)^{-\frac{1}{2}\|W_\ell\|} \geqslant (2m-1)^{-\frac{1}{2}L}\right) \\ &\leqslant (2m-1)^{\frac{1}{2}L}\,\mathbb{E}(2m-1)^{-\frac{1}{2}\|W_\ell\|}\end{aligned}$$

to get the conclusion. □

7.2. **Folding the labelled graph.** Labelling a graph by plain random words does generally not result in a reduced labelling. Nevertheless, we can always fold the resulting labelled graph. Here we show that in the circumstances needed for our applications, this folding is a quasi-isometry. This will allow a transfer of the Gr' small cancellation condition from the unfolded to the folded graph.

Proposition 7.8. *For any $\beta > 0$, for any $v \in \mathbb{N}$, for any $C \geqslant 1$, there exists an integer j_0 such that for any $j \geqslant j_0$, for any graph Γ satisfying the conditions:*

(1) *Vertices of Γ are of valency at most v.*

(2) Diameter$(\Gamma) \leqslant Cg$ *for all* i*, where* g *is the girth of* Γ*.*

then the folding map $\Gamma^j \to \overline{\Gamma^j}$ *from a random labelling* $\Gamma^j \to B$ *to the associated reduced labelling* $\overline{\Gamma^j} \looparrowright B$ *is a* $(\frac{\theta}{1-\theta}, \beta jg, gj)$ *local quasi-isometry, with probability tending to* 1 *as* $g \to \infty$*.*

We use the notation from [GdlH90] for local quasi-isometries: an (a,b,c) local quasi-isometry is a map f such that whenever $d(x,y) \leqslant c$ we have $\frac{1}{a}d(x,y) - b \leqslant d(f(x),f(y)) \leqslant ad(x,y) + b$. Here folding obviously decreases distances so that only the left inequality has to be checked.

Remark 7.9. Below we will make repeated use of the following: The number of paths of length ℓ in Γ^j is at most $j^2\, v^{Cg+\ell/j}$. Indeed, the number of points in Γ is at most v^{Cg}, and once a point is chosen the number of paths of length k originating at it is at most v^k. Now specifying a path in the subdivision Γ^j amounts to specifying a path in Γ and giving two integers between 1 and j to specify the exact endpoints.

Proof. Unwinding the definition of local quasi-isometries, we have to prove that any immersed path of length $\beta gj + \ell \leqslant gj$ in Γ^j is mapped onto a path of length at least $\frac{1-\theta}{\theta}\ell$ in $\overline{\Gamma^j}$.

By Remark 7.9, there are at most $j^2\, v^{Cg+g}$ paths of length gj in the subdivision Γ^j of Γ. Fix such a path, of length say $\beta gj + \ell$.

Since the length of the immersed path is at most $gj = \text{girth}(\Gamma^j)$, the path does not travel twice along the same edge. Consequently, the labels appearing on this path are all chosen independently. Then by Lemma 7.6, the probability that its length after folding is less than $\frac{1-\theta}{\theta}\ell$ is less than

$$(2m)^{-(1-\theta)(\ell+\beta gj)+\theta\frac{1-\theta}{\theta}\ell} = (2m)^{-(1-\theta)\beta gj}$$

for this particular path. Since the number of choices for the path is at most $j^2\, v^{Cg+g}$, if j is large enough depending on C, β and θ, namely if $v^{C+1}(2m)^{-(1-\theta)\beta j} < 1$, then the probability that *there exists* a path violating our local quasi-isometry property will tend to 0 as $g \to \infty$. □

Corollary 7.10. *In the same circumstances, the girth of* $\overline{\Gamma^j}$ *is at least* $\frac{1-\theta}{\theta} - \beta$ *times that of* Γ^j*.*

Proof. Take a simple closed path p in $\overline{\Gamma_j}$. It is the image of a non-null-homotopic closed path q in Γ^j, whose length is by definition at least $gj = \text{girth}\,\Gamma^j$. Let q' be the initial subpath of q of length gj. We can apply the local quasi-isometry statement to q', showing that its image p' has length at least $\frac{1-\theta}{\theta}gj - \beta gj$, which is thus a lower bound on the length of p. □

7.3. **Pieces in the unfolded and folded graphs.** Here we show that under the circumstances above, the probability to get a long piece in the folded graph is very small.

Suppose again that we are given a graph Γ of degree at most v, of girth g and of diameter at most Cg. Consider its j-subdivision Γ^j endowed with a random labelling and let $\overline{\Gamma^j}$ be the associated folded labelled graph.

Let p, p' be two immersed paths in $\overline{\Gamma^j}$. Let q, q' be some preimages in Γ^i of p, p'. If p and p' are labelled by the same word, then q and q' will be labelled by some freely equal words, so that pieces come from pieces.

Note that in a graph labelled by non-reduced words, there are some "trivial pieces": e.g. if some aa^{-1} appears next to a word w, then $(w, aa^{-1}w)$ will be a piece. Such pieces disappear after folding the labelled graph; this is why we discard them in the following.

Proposition 7.11. *Let q, q' be two immersed paths in a graph Δ of girth g. Suppose that q and q' have length ℓ and ℓ' respectively, with ℓ and ℓ' at most $g/2$. Endow Δ with a random labelling. Suppose that after folding the graph, the paths q and q' are mapped to distinct paths. Then the probability that q and q' are labelled by two freely equal words is at most*

$$C_{\ell,\ell'}(2m)^{-(1-\theta)(\ell+\ell')}$$

where $C_{\ell,\ell'}$ is a term growing subexponentially in $\ell + \ell'$.

Proof. Let w and w' be the words labelling q and q' respectively.

First, assume that the images of q and q' in Δ are disjoint. Then the letters making up w and w' are chosen independently, and thus the word ww'^{-1} is a plain random word. Thus is this case the proposition is just a rewriting of the definition of θ.

Second, suppose that the paths do intersect in Δ: this results in lack of independence in the choice of the letters making up w and w' (the same problem is treated in a slightly different setting in [Oll04], section "Elimination of doublets"), which needs to be treated carefully. Since the length of these words is less than half the girth, the intersection in Δ is connected and we can write $w = u_1u_2u_3$, $w' = u'_1u_2u'_3$ where the u_i's are *independently chosen* random words (depending on relative orientation of w and w', u_2^{-1} rather than u_2 may appear in w'). We can suppose that $u'_1u_1^{-1}$ is not freely trivial: otherwise the two paths start at the same point after folding, and so if $w = w'$ we also have $u'_3u_3^{-1} = e$ so that they also end at the same point after folding, but this is discarded by assumption. Likewise $u'_3u_3^{-1}$ is not freely trivial.

Let v_1, v_2, v_3, v'_1, v'_3 be the reduced words freely equal to u_1, u_2... respectively.

Lemma 7.5 tells us that the words v_1, v_2... are random reduced words. Now let us draw a picture expressing the equality $v_1v_2v_3 = v'_1v_2v'_3$:

Note that the two copies of v_2 have to be shifted relatively to each other, otherwise this means that $u'_1u_1^{-1}$ and $u'_3u_3^{-1}$ are freely trivial.

Let k be the length shared between the two copies of v_2. Now let us evaluate the probability of this situation knowing all the lengths of the words v_1, v_2, ... Conditionnally to their lengths, these words are uniformly chosen random reduced words by Lemma 7.5.

We begin with the two copies of v_2: though they are not chosen independently, since we know that they are shifted, adding letter after letter we see that the probability that they can glue along a subpath of length k is a

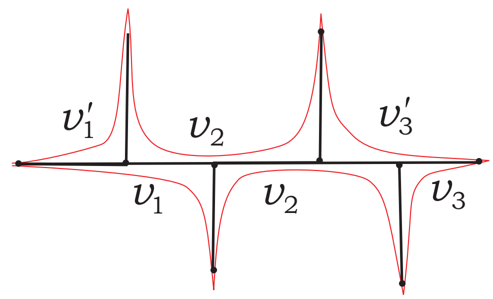

FIGURE 3.

most $1/(2m-1)^k$. Once v_2 is given, the words v_1, v_3, v_1', v_3' are all chosen independently of each other. The probability that they glue according to the picture is $1/(2m-1)^{L-k}$ where L is the total length of the picture. So the overall probability of such a gluing is $1/(2m-1)^L$.

We obviously have $\ell + \ell' = |v_1| + 2|v_2| + |v_3| + |v_1'| + |v_3'| = 2L$. Now by Proposition 7.7 applied to all these words separately, the probability of achieving this value of $|v_1| + 2|v_2| + |v_3| + |v_1'| + |v_3'|$ is less than

$$C_{\ell,\ell'}\,(2m)^{-(1-\theta)(|u_1|+2|u_2|+|u_3|+|u_1'|+|u_3'|)}(2m-1)^{\frac{1}{2}\,2L} = C_{\ell,\ell'}\,(2m)^{-(1-\theta)(\ell+\ell')}(2m-1)^L$$

where $C_{\ell,\ell'}$ is a term growing subexponentially in $\ell+\ell'$.

So the overall probability of such a situation, taking into account the possibilities for L between 0 and $\ell+\ell'$, is at most

$$\sum_{0\leqslant L\leqslant \ell+\ell'} (2m-1)^{-L}\,C_{\ell+\ell'}\,(2m)^{-(1-\theta)(\ell+\ell')}(2m-1)^L$$

since we just proved above that $(2m-1)^{-L}$ is an upper bound for the probability of the situation knowing L. But this is equal to $C'_{\ell+\ell'}\,(2m)^{-(1-\theta)(\ell+\ell')}$ where $C'_{\ell+\ell'}$ is another term growing subexponentially in $\ell+\ell'$. □

We are now ready to prove Proposition 7.4 stating that the $Gr'(\alpha)$ condition is satisfied with overwhelming probability. In order to avoid heavy notations, we will directly prove the stronger variant of the Gr' condition involving the girth instead of the length of cycles containing the pieces (see section 2).

Proof of Proposition 7.4, small cancellation part. Since ruling out small pieces rules out larger pieces as well, it is enough to work for small α.

Let $\bar{g}$ be the girth of $\overline{\Gamma^j}$. By Corollary 7.10, we can assume that $\bar{g} \geqslant \left(\frac{1-\theta}{\theta} - \beta\right) gj$ with overwhelming probability, for arbitrarily small β.

Let p, p' be two distinct immersed paths in $\overline{\Gamma^j}$ forming a α-piece; both p and p' are of length $\alpha\bar{g}$. Let q and q' be some immersed paths in Γ^j mapping to p and p'.

Suppose that the length of q (or q') is greater than $gj/2$. By applying the local quasi-isometry property to an initial subpath of q of length $gj/2$ we

get that the length of p would be at least $\frac{1-\theta}{\theta}gj/2 - \beta gj$. But the length of p is exactly $\alpha\bar{g} \leqslant \alpha gj$, so that if α and β are taken small enough (depending on θ) we get a contradiction. Hence, the length of q is at most $gj/2$, so that we are in a position to apply Proposition 7.11.

The length of q and q' is at least that of p and p' namely $\alpha\bar{g}$, and since $\bar{g} \geqslant \left(\frac{1-\theta}{\theta} - \beta\right) gj$, q and q' form a $\alpha\left(\frac{1-\theta}{\theta} - \beta\right)$-piece in Γ^j. Now Proposition 7.11 states that for fixed q and q' in Γ^j, the probability of this is at most $C_{gj}(2m)^{-(1-\theta)2gj\alpha\left(\frac{1-\theta}{\theta}-\beta\right)}$, where C_{gj} is a subexponential term in $|q| + |q'| \leqslant gj$.

By Remark 7.9, the number of choices for q and q' is at most $j^4\, v^{(2C+1)g}$. So the probability that one of these choices gives rise to a piece is at most

$$j^4\, v^{(2C+1)g}\, C_{gj}(2m)^{-(1-\theta)2gj\alpha\left(\frac{1-\theta}{\theta}-\beta\right)}$$

Now, if β is taken small enough (depending only on θ) and if j is taken large enough (depending on α, θ and C but not on g), namely if

$$v^{2C+1}\,(2m)^{-(1-\theta)2j\alpha\left(\frac{1-\theta}{\theta}-\beta\right)} < 1$$

then this tends to 0 when g tends to infinity. □

Proof of Proposition 7.4, missing word part. We now prove that for any $\alpha' > 0$, in the same circumstances, there exists a reduced word of length $\alpha'\,\mathrm{girth}(\overline{\Gamma^j})$ not appearing on any path in $\overline{\Gamma^j}$.

Let p be a simple path of length $\alpha'\bar{g}$ in $\overline{\Gamma^j}$. It is the image of some path q in Γ^j of length at least $\alpha'\bar{g} \geqslant \alpha'(\frac{1-\theta}{\theta} - \beta)gj$. But by Remark 7.9, the number of such paths in Γ^j is at most $j^2\, v^{Cg+\alpha'(\frac{1-\theta}{\theta}-\beta)g}$, whereas the total number of reduced words of this length is at least $(2m-1)^{\alpha'(\frac{1-\theta}{\theta}-\beta)gj}$. So if j is taken large enough (depending on α' and θ but not on g) that is if

$$v^{C+\alpha'(\frac{1-\theta}{\theta}-\beta)} < (2m-1)^{\alpha'(\frac{1-\theta}{\theta}-\beta)j}$$

then the possible reduced words outnumber the paths in $\overline{\Gamma^j}$ when $g \to \infty$, so that there has to be a missing word. □

8. Problems

Does there exist a finitely presented group N with property T such that $\mathrm{Out}(N)$ is infinite?

Let Q be a f.p. group with property T. Does there exist word-hyperbolic G with property T and f.g. normal subgroup N such that $Q = G/N$?

Do there exist f.p. Kazhdan groups which are not Hopfian or coHopfian?

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